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A Hybrid Analytical Model of Growth Regulation in Animals

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Abstract

The aim of this study was to build and analyse a model of ontogenetic growth of animals. The model was built based on experimental data and field observations. The growth of pigs was modelled by a non-local hybrid technique. This technique treats time as a discrete variable. In this study the growth of pigs was modelled beginning the stage of the rapid growth up to the maximum weight. The growth was modelled as a dynamic system. It was shown that the trajectory of the growth is neither smooth nor continuous. The main theme in this study is transition to a new growth phenotype. There are two results in this study. At a certain point in animal's ontogeny the trajectory of the growth undergoes a first-order phase transition. In the next stage, during bifurcation, new trajectories of growth emerge; this sequence of events has a biological meaning. The emerged trajectories differ from the initial trajectory, and from each other in essence. In the model, one trajectory of the growth emerges instantly. For other growth trajectory to emerge it takes half a year. In a population of animals, it is a general situation. Individual animals can take on only one of the emerged trajectories. In this study, a two-stage process of a transition to a new ontogenetic trajectory or a new phenotype was revealed. The transition to the new growth phenotype is to consider as the model of the pattern of the systemic regulation of growth.

Keywords: quantitative trait, growth phenotype, ontogenetic trajectory, dynamic system, fold bifurcation, system analyses

1 Introduction

The problem this study deals with is ontogenetic growth of animals. A problem in biology is how an organism makes out that its final size and weight is reached. As a result, the growth stops (Lui and Baron., 2011). A part of the problem has been considered in this study. The aim of this study was to build and analyse an analytical model of growth of pigs. In wildlife, animals compete with each other to survive and reproduce. In this kind of competition weight and size are essential if not decisive. How the species-specific weight and size are determined in animals' ontogeny remain unclear. And, why the growth rate between mammal species differs many folds is less well understood. In animals, the growth is a dynamic process. In this study the growth of animals was modelled and analysed as a dynamic system. There are two main results in the study. First result suggests that feed conversion coefficient Z is the order parameter. Second result implies that the dynamics of Z exerts systemic effects on the regulation of growth. We shall discuss the advance made in modelling the growth of animals and how this model could form part of a larger concept. The novelty of this study is a concept of the systemic regulation of growth. The concept entails a new model of the transition from one growth phenotype to one more in ontogeny.

1.1 Growth and Growth Control in Animals

Growth of animals is an intricate process that is linked to development to form adults with proper size and proportions. In animals, genetics is a determining factor of growth (Boulan et al., 2015). In animals and humans growth rate is considered as a quantitative trait. In this research field the trait is thought to be determined by both genetic and environmental factors (Texada et al, 2020). The genetic factors one can consider as the growth phenotypes. In this study we model only one environmental factor, and it is feed; other environmental factors are considered to be optimal. For animals' growth, nutrition is the main environmental factor in the determination of body size. In animals, nutritional information is conveyed at both the cellular and the systemic levels to provide balanced growth (Boulan et al., 2015). In the course of growth, the balance between weight gain and loss is determined by the quantity and quality of food consumed (Kelly et. al., 2021).

In an animal's ontogeny, its growth is under systemic control. That growth is under tight control is supported by the precision observed in the sizes of organisms and their parts (Lander., 2011). There are a few factors involved in the regulation of growth (Penzo-Méndez and Stanger, 2015). Reportedly, some of the factors are the following: morphogens, growth factors, mechanical force, hormones, and systemic factors (Lander., 2011). That hormones are involved in regulation of growth is well known (Lui and Baron., 2011). However, hormones are not the main factors in regulation of growth. Moreover, the same hormones that regulate organ growth also regulate organ patterning (Mirth and Shingleton., 2019). In animal organisms one can suggest existence of the mechanisms that simultaneously scale pattern to growth, and growth to pattern. This kind of growth control is supposed to exert unknown systemic factors (Vollmer et al. 2017). The mechanism of this scaling most probably is feedback loops. The termination of growth at a certain time points in ontogeny is thought to be under the same control. In animals, the feedback loops of growth regulation provide the nonlinear dynamics (Mirth and Shingleton, 2019). In individual animals this kind of growth control results in a species-specific weight and size.

1.2 Methodological Aspects of the Model

The concept of the model is as follows. Animals and humans are thermodynamically open systems. To sustain life, they need food or feed. For animals, feed is the environmental factor. In animals, growth and development are possible if feed is available in sufficient quantity and quality. In the model the current weight M of an animal is a function of a feed consumed F. The feed consumed F has been transformed by an organism to an animal weight M with efficiency Z. In the model, the feed efficiency Z is a variable that models how efficiently the feed consumed F has been converted to current weight M. In this study the growth invariant K has been used. The growth invariant K is a species-specific parameter; it is related solely to domestic pigs. In development of animals there are two known rules. Development unfolds in stages; the next stage starts only if the preceding stage completed. And, in an organism, from the very beginning of growth the result is predetermined to a species-specific form and weight. Under these conditions the dynamic of growth was modelled. Under these conditions, a growing organism was considered as a dissipative system. In living beings, the conditions for dissipative structures are met; they are open systems, governed by nonlinear relations, and function far from thermodynamic equilibrium (Goldbeter.,2018). In this aspect the two following opinions are important. In ontogeny, growth of individual animals has been controlled by a series of feedback loops. The feedback loops are the cause of nonlinear processes in the organisms (Mirth and Shingleton, 2019). In living beings, nonlinearity of the growth processes is a precondition for the emergence of dissipative structures. In this study, findings have been discussed bearing in mind the above theoretical notions.

2 Methods

In this study, methods of mathematical modelling were used. A non-local hybrid model (Stass, 2022) of animals' growth was built based on experimental data and field observations. The model of the growth regulation in animals was built by extension of an earlier introduced model (Stass, 2022). To build the model of ontogenetic growth a hybrid modelling technique was used. The hybrid technique combines both discrete and continuum variables and methods. In the model, the current time t was considered as a discrete variable. Minimum time span in the model, Δt is one day; this period of time corresponds to one cycle of the circadian rhythm. In animals, the circadian rhythm is associated with cycles of growth and development (Kelly et. al., 2021). In this study some theoretical ideas were used to explain results but nether to build the model. It is by analysing the model both biological notions of the growth and mathematical abstractions were used.

3. Results

In this study we have built a nonlinear hybrid model of pigs' growth. The model describes a few aspects of ontogenetic growth and pays special attention to the qualitative change in the trajectory of the growth. At that point, bifurcation of the growth trajectory takes place. As a result, new growth phenotypes emerge.

3.1 A Hybrid Model of Growth of Pigs

In this study, the experimental data analyses have led to the following system, given by

$$\begin{cases} \frac{M}{m_o} = 2K - 1 + \frac{(Z-2K)(K-1)}{ZK} \\ K = \frac{Mt}{m_o(2t-t_o)} \end{cases} , \tag{1}$$

where M denotes current weight, initial weight m_o=30 kg. t denotes time from birth measured in days, initial time t_o= 90 days. K denotes a growth invariant, and Z denotes feed conversion coefficient. From (1) it follows a dynamic system, given below.

$$\frac{1}{m_o} \cdot \frac{\Delta M}{\Delta t} = \frac{K}{t} \cdot \frac{Z(2K+1)-2K}{Z(K+1)-2K} \ , \ Z > 0, \ Z \neq 1 \, . \tag{2}$$

$$\frac{\Delta K}{\Delta t} = \frac{1}{t} \cdot \frac{ZK^2}{Z(K+1)-2K} \ , \ Z > 0 \, , \ Z \neq 1 \, . \tag{3}$$

From the dynamic system (2) and (3) by excluding time t we get

$$\frac{1}{m_o}\cdot\frac{\Delta M}{\Delta K}=\frac{2K+1}{K}-\frac{2}{Z}\ ,\ Z>0\,. \tag{4}$$

It follows from (2), (3) and (4) that Z >0. This is a biologically meaningful result. The two following constrains are important for the further analyses: in the point (Z = 1)⋀(K = 1), equations (2) and (3) are discontinuous; in the point Z = 2/3 growth rate, equation (2) is equal to zero. One can infer that the trajectory of the growth is neither smooth nor continuous.

In this section we analysed variable Z. If feed is in sufficient quantity and quality then animals can grow under condition Z > 2/3. In the point Z = 2/3 animals cannot grow. Let us consider growth in the range 2/3 > Z > 0. In this interval global minimum of Z is expected. In the point of global minimum of Z, it is thought animals cannot sustain life. In this point the loss of weight reaches a degree that sustain life is impossible. It is a destruction of an organism.

In the model we found a scheme how to find global minimum of variable Z. It is as follows. Let us consider growth rate, given below.

$$\frac{1}{m_o}\cdot\frac{\Delta M}{\Delta t}=\frac{1}{t}\cdot\frac{m_o[Z[2K+1]-2K]}{m_o[Z(2K+1)-2K]-MZ}\quad . \tag{5}$$

Equation (5) follows from the system (1). Maximum growth rate in equation (5) is attainable under condition given by

$$\frac{M}{m_o}=2K+1-\frac{2K}{Z}\quad . \tag{6}$$

In equation (6) variable Z is minimum Z. It follows from the equation (5). In other words, in equation (5) maximum growth rate is attainable under condition (6) where Z is minimum value. In the next stage we will find minimum Z. To complete the task, we consider growth up to weight M_x = 600 kg. Accordingly, we consider K = K_x. Numerically K_x = 10, 196152 (Stass, 2022). It follows

$$\frac{M_x}{m_o}=2K_x-\frac{4}{K_x}\ . \tag{7}$$

From (6) and (7) we can find minimum Z, given by

$$Z_{mi}=\frac{2K^2}{K+4}\ \ , \tag{8}$$

where Z_{mi} denotes minimum Z under condition K >1. For example, for an animal in weight M_x, accordingly K = K_x, and Z_{mi} = 14,64. This result refers to the weight range from *m* up to M_x under the fastest growth rate. And, in (8) under condition K → K_o we have

$$Z_{gi}={}^{2}/_{5}\ , \tag{9}$$

where Z_{gi} denotes global minimum of variable Z. From equation (2) it follows that growth is positive if growth rate > 0 under condition Z > 2/3. For pigs in the interval 2/3 > Z > 2/5 weight loss is unavoidable.

3.2 Growth Trajectory

The dynamic system (2) and (3) specifies growth of pigs in weight range $m_o \le M \le M_x$. In this study we concentrate mainly on the growth dynamics close to the point M = M_x. I can remind the interested reader that under the model conditions M_x = 600 kg. Accordingly, $K_{|M=M_x}=K_x$, $Z_{|M=M_x}=Z_x$, $t_{|M=M_x}=t_x$. To find K_x, let us consider the limit (M →M_x) given by

$$\lim_{M\to M_x}\frac{2K-1}{K+1}=\sqrt{3}\,. \tag{10}$$

The limit (10) is based on the experimental fact. It follows from (10), K_x = 10, 196152. From equation (4) under condition M = M_x and K = K_x it follows Z_x = 62, 5102. Analytically, Z_x is given by

$$Z_x=\frac{2K_x(K_x-1)}{3}\ . \tag{11}$$

In the point (M_x, K_x, Z_x) the growth trajectory losses its stability as Z_x grows into infinity $Z_x \to \infty$. As a result, a first-order phase transition and subsequent bifurcation show up (Stass, 2022). In this section we will analyse this bifurcation. In the dynamics of the growth, bifurcation is a qualitative change. The reason the abstraction $Z_x \rightarrow \infty$ was used is the following. Under the model conditions pigs reach their maximum weight in the point $M = M_x$. In weight M_x pigs do not grow any more. This means that not any quantity of feed can increase their weight. Under the above conditions, abstraction $Z_x \rightarrow \infty$ is feasible. Before to consider this bifurcation, we shall remind some previous results (Stass, 2022). In the bifurcation point, $M = M_x$. In this point, in the instant of bifurcation, variable Z changes as follows: $Z_x \rightarrow \infty \rightarrow Z_{xv}$; where Z_{xv} denotes Z an instant after bifurcation. Numerically, Z_{xv} = 69,3076. Analytically, Z_{xv} is given by

$$Z_{xv} = \frac{2K_x^2}{3} \ . \tag{12}$$

From (11) and (12) it follows

$$Z_{xv} - Z_x = \frac{2K_x}{3} \ . \tag{13}$$

In the bifurcation point (M_x, K_x) the following equation holds.

$$K_x^2 - K_x \cdot \frac{M_x}{2m_o} - 2 = 0 \ . \tag{14}$$

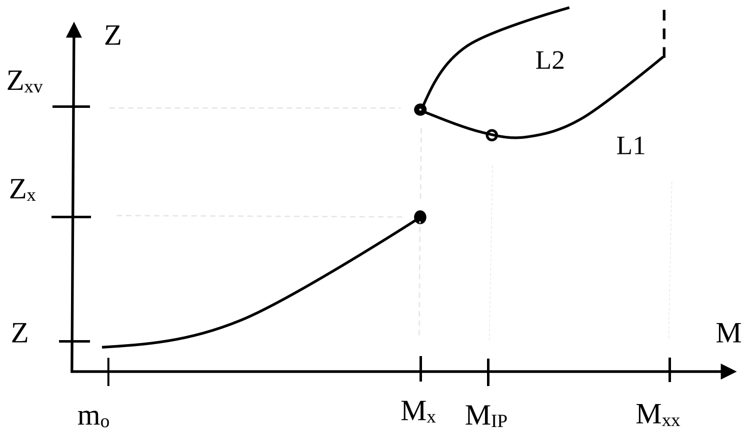

Figure 1. A first-order phase transition, and bifurcation in animals' ontogeny

A first-order phase transition point (M_x, Z_x) • bifurcation point (M_x, Z_{xv}). **o** inflection point

Transition $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_x)$ corresponds trajectory L1

]Transition $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_{xv})$ corresponds trajectory L2

It follows from (14) that on this trajectory there is an inflection point. $M_{IP} = 2m_oK_x$, where M_{IP} denotes the inflection point. In this inflection point growth rate starts to increase. In brief notation this trajectory is given by $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_x)$. Figure 1 shows this transition as a $(m_o \rightarrow M_x) \rightarrow$ L1. Trajectory L1 is the new growth phenotype. The main point to note is that in this case, during the phase transition, K_x remain unchanged. It follows this bifurcation trajectory emerges in an instant; this transition does not take time. In the study, the phase transition precedes bifurcation; this sequence of events has biological meaning.

Let us consider second emerged trajectory in this bifurcation. Figure 1 shows this trajectory marked L2. In short notation this trajectory is given by $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_{xv})$. This transition takes half a year. However, in both cases weight M = M_x. In both cases bifurcation point of variable Z is Z = Z_{xv}, figure 1. Let us consider equation (4). This equation is the most general in the model. It follows from the dynamic system (2) and (3). To find K_{xv} one has to substitute M_x and Z_{xv} into equation (4). It follows K_{xv} = 10, 181583. With this result one can find how long does transition $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_{xv})$ take. In the point (M_x, K_x), animals age one can find from the system (1). Under condition ($M = M_x$, $K = K_x$), it follows t_x = 6,408 years. In the same way, under condition ($M = M_x$, $K = K_{xv}$), it follows t_{xv} = 6,912 years. The difference t_{xv} - t_x = 0,504 years or 184 days. This time is needed for an animal in weight M_x to left one growth trajectory and take on another. It is to complete the following transition $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_{xv})$. Figure 1 displays this transition as a $(m_o \rightarrow M_x) \rightarrow$ L2. Trajectory L2 is the new growth phenotype.

A functional relation between variables is given by

$$\frac{\Delta K_{xv}}{\Delta K_x} = 1 - \frac{2K_x^2}{3Z_x^2\,(K_x+1)} \quad . \tag{15}$$

Let us consider limit ($Z_x \to \infty$) of (15), given below.

$$\lim_{Z_x \to \infty} \frac{\Delta K_{xv}}{\Delta K_x} = 1\ . \tag{16}$$

From the limit (16) it follows that in an instant of bifurcation K_x, and K_{xv} are indistinguishable. After an instant, new growth trajectories emerge. One can conclude that after bifurcation, K_x, and K_{xv} provide boundary conditions for the new growth trajectories.

4. Discussion

In this section a few abstractions of the systems analyses were used. The physical world is mostly nonlinear. This is a warning against making assumptions based on traditional linear thinking (Stewart, 2011). To avoid mistakes which stem from phenomenological approach mathematical models have been used. Another reason for using models is a wish to understand biological processes. In biology and medicine, data do not translate freely into understanding, let alone treatment. We need mathematical models to deliver interpretation of data, and understanding of the studied processes (Coveney et al. 2016). In this study, the mathematical model of the growth of pigs provides advanced insight into a few aspects of the growth dynamics. It follows from the model that growth is to consider as a dynamic system. And, in the course of the growth the main qualitative events may be explained by systems analyses theory. In this model, feed conversion efficiency Z is the order parameter. In the model, the dynamics of Z causes both a first-order phase transition and subsequent bifurcation. This sequence of events in an organism is the necessary precondition for transition from the current to a new growth trajectory. The onset of a new growth phenotype is main theme in this study. In individual animals variable Z is to consider as a systemic factor in the growth regulation mechanisms. In this study we have carried out a basic analytical analysis of the model. A computer analyses of the model may reveal more details.

4.1 Qualitative Events in the Growth Dynamics

Systemic growth control ensures that organs grow in correct proportion to each other and to the entire organism (Texada et al, 2020). In this study we looked into one aspect of the systemic regulation of growth. This aspect is the growth trajectory bifurcation. At a certain point in ontogeny growth stops. In this study, we have built a model of how the growth stops and resumes in ontogeny. Figure 2 shows this process in general.

Bifurcations of a trajectory of dynamic system results in a qualitative change in its course (Roesch and Stumpf, 2019). One can distinct the two kinds of the changes: smooth, and catastrophic or discontinuous. The both kinds describe the different dynamics of modelled systems. The catastrophic change includes a discontinuity in a trajectory thus giving place to a first-order phase transition (Sardanyés et al., 2018). A first-order phase transition displays a sudden, discontinuous change in the order parameter at the critical point (Heffern et al., 2021). Figure 1 displays such change. Figure 2 displays bifurcation of the growth trajectory in general. In the study the phase transition presides bifurcation; this sequence of events has a biological meaning.

In this section we will analyse qualitative events in the dynamics of growth of animals. We will carry out analyses in stages to show the biological meaning of each stage. It is an experimental fact that a boar reached its maximum weight M_x = 600 kg. It is an experimental fact that the boar in weight M_x was 6,40 years old. In the study, the mentioned facts this model describes analytically. Under the model conditions, we analyse events that take place when an animal reaches its maximum weight. The events follow from the analyses of the model.

In weight M_x a pig reaches its maximum weight. In this point the animal does not grow any more. In this point its growth stops. It has the consequence that at the point M_x feed conversion coefficient Z grows into infinity ($Z_x \to \infty$). This means that not any quantity of feed can increase this animal's weight M_x. This has the following consequences. At the point (M_x, K_x, Z_x) a first-order phase transition ($Z_x \to \infty \to Z_{xv}$) occurs (Stass, 2022). We will show this event in the following steps $Z \to Z_x$, then $Z_x \to \infty$, then $\infty \to Z_{xv}$, and then $Z_{xv} \to Z_{IP}$, where Z_{xv} denotes the feed conversion coefficient Z an instant after the phase transition, and Z_{IP} denotes Z in inflection point. The same changes are shown in terms of the growth invariant K; $K \to K_x \to K_{xv} \to K_{IP} \to K_{xx}$. The growth trajectory advances from a point (M, Z, K) to the point (M_x, Z_x, K_x). In this point, a first-order phase transition occurs. This phase transition occurs due to the dynamics of Z. Initially stable growth trajectory $m_o \to M$, at the point M_x loses its stability when growth stops and feed conversion coefficient Z grows into infinity $Z_x \to \infty$. As a result, the following setup follows. In the point (M_x, Z_x, K_x) an animal was 6, 40 years old and 600 kg in weight; its longevity was uncertain. In the point (M_x, $Z = \infty$) it acquires some additional qualities. In this point, growth invariant K changes from K_x to K_1 ($K_x \rightarrow K_x \rightarrow K_1$) (Stass, 2019., 2022). As a result, longevity of animals that reached their maximum weight M_x, and underwent transition $Z_x \rightarrow \infty$ and remain in this position is 23, 90 years. This follows from the calculation under condition $(M=M_x) \wedge (K=K_1)$,(Stass, 2019). The animals took on a stable growth trajectory; in this case with the growth rate zero. On this trajectory $(M=M_x) \wedge (K=K_1) \wedge (Z=\infty)$.

The next move is for the animals that can grow further and potentially reach the species maximum weight M_{xx}. The animals that can grow beyond M_x, resume the growth in the point (M_x, Z_{xv}, K_x) or in the point (M_x, Z_{xv}, K_{xv}). These two points we will consider separately. The transition to the point occurs as follows ($Z_x \rightarrow \infty \rightarrow Z_{xv}$) and, in one case $K_x \rightarrow K_x$, in other $K_x \rightarrow K_{xv}$. At first, let us consider transition $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_x)$. This transition occurs instantly, it does not take time. As a result, a new growth trajectory emerges. This is a mathematical abstraction of a quick switch on a new emerged trajectory. On figure 1 this trajectory is marked L1. On this trajectory there is an inflection point, denoted (M_{IP}, Z_{IP}); it follows from the equation (6). In this inflection point growth rate starts to increase. It takes 89 days for a boar to get in this point from point (M_x, K_x, Z_{xv}). This follows from the calculation of time difference between K_x, and K_{IP} considering transition $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_x)$. This trajectory was analysed by Stass (2022).

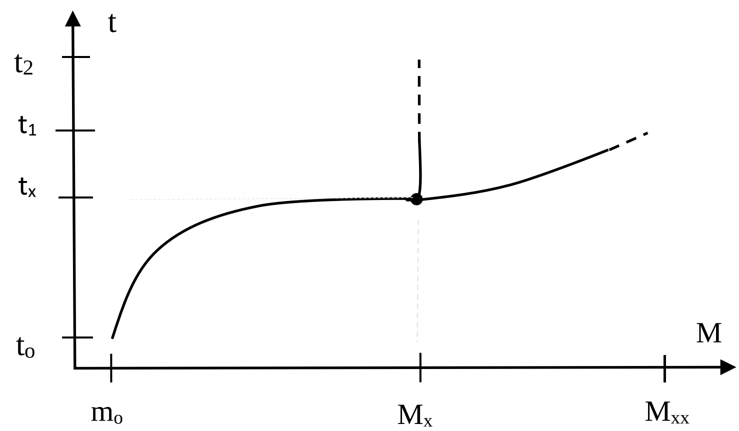

Figure 2. Growth trajectory bifurcation in animals' ontogeny

• bifurcation point

Other growth trajectory emerges in the same point as a result of the following transition $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_{xv})$. This transition takes half a year. Figure 1 displays this transition as the trajectory $(m_o \rightarrow M_x) \rightarrow$ L2. We discuss it below. A necessary precondition for an animal to take on one of the emerged trajectories is its capacity to accumulate weight M_x. Individual maximum weight M_x cannot put on all animals. Many animals cease to grow not reaching their maximum weight M_x. Figure 2 displays a general situation in a population depicting growth trajectory bifurcation. Individual animals can take on only one trajectory; in this case either $(m_o \rightarrow M_x) \rightarrow$ L1, or $(m_o \rightarrow M_x) \rightarrow$ L2, figure 1. The emerged trajectories or phenotypes L1, and L2 differ from each other in essence. As a result, one may suggest genetic determination of the transitions. Let us compare the above trajectories. Both trajectories emerge in the same point $Z = Z_{xv}$. In this point, in both cases animal's current weight $M = M_x$. In both cases the only difference is growth invariant K; in one case it is K_x in other K_{xv}. This has the following consequences. Transition $(m_o \rightarrow M_x) \rightarrow$ L1 takes place in an instant. One could call it a quick switch. This transition could serve as a model of genetic switch. By contrast transition $(m_o \rightarrow M_x) \rightarrow$ L2 takes half a year. One could call it a physiological switch. In both cases, variable Z is the order parameter. There is procedure for genes L1, and L2 to emerge at once as well, and produce heterozygote L12. However, figure 1 is incomplete. Figure 1 shows bifurcation, though there should be depicted trifurcation. One more trajectory, namely $(Z_x \rightarrow \infty) \wedge (K_x \rightarrow K_x)$ was not displayed. On this trajectory animals do not grow. On this trajectory animals weight remain constant ($M = M_x$, $Z = \infty$). Trajectory ($M = M_x$, $Z = \infty$) is the new growth phenotype; we can denote it L3. In other words, trifurcation has the following trajectories: $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_x)$, $(Z_x \rightarrow \infty \rightarrow Z_{xv}) \wedge (K_x \rightarrow K_{xv})$, and $(Z_x \rightarrow \infty) \wedge (K_x \rightarrow K_x)$, or L1, L2, and L3, respectively. On all trajectories in trifurcation point $M = M_x$. The further dynamic of growth determine variables Z and K. If to compare this bifurcation with pitchfork bifurcation than one can note a considerable difference. For the normal form pitchfork bifurcation, both emerged trajectories show up simultaneously and they are symmetric. In this model this is not the case. It is understandable, this is a non-local model. There are no standard methods to analyse non-local dynamic systems. Though, this kind of dynamic models has uncommon features. In ontogeny, in the point (M_x, Z_x, K_x) a first-order phase transition takes place ($Z_x \rightarrow \infty \rightarrow Z_{xv}$). As a result, in the point (M_x, Z_{xv}, K_x) bifurcation, or in the point (M_x, Z_{xv}, K_{xv}) trifurcation takes place. And, in the same point and simultaneously, bifurcation of the parameter K takes place (Stass, 2022).

It follows that this phase transition precedes both bifurcations. The events are complex and intricate. Consider that the growth rate is a function of M, Z, and K. Since we consider growth of animals, the phase transition takes in whole organism, with all its subsystems and scales. A possible explanation is the following. In the point (M_x, Z_x, K_x) a growing animal as a whole organism reaches biological competence for a new stage in development. As a result, instability in the form of a first-order phase transition develops. As a result, bifurcation shows up and new growth trajectories emerge. In other words, it is a model of transition to a new ontogenetic trajectory or a new stage in the growth and development. This transition is discontinuous, and unfolds as sequence of two events. In one case it happens instantly in other it takes half a year. In both cases though an animal takes on a new growth trajectory.

In ontogeny, this sequence of events can serve as a model of a pattern of systemic regulation of growth in individual animals. In the course of the growth, at the point of a first-order phase transition an organism is set or conditioned for ontogenetic change. The first-order phase transition provides that an organism as the whole is taken in. The organism is set out to transition. At the point of the phase transition the organism has reached its competence for the change. In the course of the phase transition the change occurs. During subsequent bifurcation this change or transition is accomplished; it takes place and takes shape. As a result, the transition turned into a new and irreversible trajectory of growth. The new growth phenotype set in. In the model the following three growth phenotypes emerge: L1, L2, and L3. In individual animals, there is procedure for the genes L1, and L2 to express simultaneously and coherently, and produce a distinct phenotype L12. In other words, on condition that L1, and L2 genes express at once, at that time the heterozygote phenotype L12 is produced. In this model, the heterozygote L12 is a coherent whole of L1, and L2. This is in line with the classical genetics rule, which defines formation of a heterozygote. In this model, the rule finds explanation for the quantitative trait. In the study, the development of the phenotype L12 is certainly feasible; the genetic change takes place and then physiological transition follows. It is feasible to speculate that the heterozygote phenotype L12 can emerge in the point (K_{xv}, Z_{xv}) as a limit cycle. In other words, in the way like the Hopf bifurcation.

In this model, it is possible formulate generalisation of the results. In a mature animal, after transition to a new growth trajectory one of the three following processes develops. A physiological change with a limited genetic involvement. A genetic switch with a limited physiological consequence. A coherent whole of the both former processes.

This model is an example of the parametric regulation of growth. In this model, initial conditions do not influence final result. It is essential in this model that parameter K, the growth invariant, is dimensionless. In this study parameter K provides boundary conditions for the dynamic system of growth.

In this study, we can give a reading of the growth trajectory. For continuum systems, analyses in two dimensions can differ from results in one dimension. In three dimensions, the picture may be completely different. Under the model conditions, we can give the following interpretation. In one dimension, the growth trajectory, figure 1, can be considered as fold bifurcation, curves L1 and L2. In two dimensions, the same bifurcation of the same trajectory, figure 1, can be considered as focus bifurcation. Bifurcation of the weight gain trajectory, figure 2, is thought to be of pitchfork form and of supercritical type. In this interpretation the emerged trajectories L1, L2, L12, and L3 were considered stable.

In this section we will assess the extent to which this model is applicable. The model was built based on experimental data. It is feasible to suggest that it can be extended to other processes associated with growth. In other words, the model is applicable to a wider range of processes. In circumstances where the initial process stops for a while to give place the next stage processes this model may be used. For example, it can be applied to modelling some aspects of morphogenesis. Or it may be used to model some stages of cancer growth. In other words, the model is applicable not only to the whole organism but also to its parts. It may perhaps be helpful to emphasise that the model is applicable in circumstances where growth stops for a while, and stops for different reasons. This is to say not only because the individual maximum weight was reached, as it was the case in this study. In humans and animals' ontogeny, there are periods of no growth (Stass, 2021). Right in these circumstances the model may be used.

4.2 The Trajectory of Growth

We can remind the interested reader an empirical notion of variable Z. Feed conversion coefficient Z is an intricate trait. Its linear and common logic is the following. If $Z = 1$ then all feed consumed is converted to an animal's body weight. If $Z > 1$ and feed is in sufficient quantity and quality then animals can grow. If $0 < Z \leq 1$ animals cannot grow. The meaning of the condition $Z \leq 1$ is that to sustain life organism must catabolise own tissues to maintain functions, and suffer weight loss. The model says that the nonlinear dynamics of variable Z is more complicated. It is discussed below.

One can infer from the model that the trajectory of the growth is neither smooth nor continuous. Under the model conditions, global minimum of variable $Z = 2/5$. In this point it is thought sustain life is impossible. However, if to look at this result the other way round, then one can suggest that under condition Z > 2/5 sustain life is possible. And, in the point $Z \geq 2/3$ growth starts. In the point $Z = \infty$ the growth stops. In the two following points the growth trajectory is discontinuous: Z = 1, and $Z = \infty$. These two points require special attention. In the point $Z = \infty$ animals do not grow. In the point Z = 1 growth rate is uncertain. Growing individual animals have at least one local growth maximum. In individual animals, there are local growth maximum during the stage of the rapid growth. After this stage in weight of about 100 kg domestic pigs reach puberty. In this weight the growth rate slows down and level off. The dynamics of the further growth as well as longevity of animals are contingent on their ability to reach individual maximum weight.

In this model, food conversion coefficient Z is feasible to consider as an aggregated variable of flow of metabolites. The dynamics of the flow influence the growth and development. In morphogenesis, certain components of the flow may act as morphogens. In ontogenesis, flow of some metabolites can act as systemic growth control factors. In both cases flow of certain metabolites is to consider as systemic factors in the growth regulation of animals. In both cases genetic determination of growth is mediated by flow of the metabolites. The metabolites have been produced by certain ferments in the quantities to establish control of growth by feedback loops. In this way the genetic control of growth could be understood. The novelty of this reading is a concept that it is the food conversion coefficient Z, which sums up processes of the systemic growth control on organism scale.

4.3 Growth Trajectory Bifurcation Is a Qualitative Transition in Ontogeny

In a population of animals, the dynamics of growth with bifurcation in individual animals, creates three sets with the growth phenotypes. There is one set with the animals that can reach their individual maximum weight at the bifurcation point. The second set with the animals which can grow beyond the bifurcation point to reach the species maximum weight. And in the third set, there are animals which can neither reach their individual maximum weight nor reach bifurcation point. The three sets one can translate in terms of phenotypes. One set with the phenotypes, which cannot reach the bifurcation point. Second set with the phenotypes, which can reach the bifurcation point but do not grow further. Third set with the phenotypes, which can pass through the bifurcation point and continue to grow to reach the species maximum weight M_{xx}. In a population, the three sets with the asymptotic growth phenotypes considerably differ from each other. In each of the sets animals' performance and longevity is not the same. The main difference between the phenotypes in the three sets are growth rate, and life span.

4.4 Cessation of Growth in Animals

In the model, there are two similar patterns of the growth cessation in animals. In both cases growth stops due to the dynamics of Z; namely Z_x, or Z_{xx} grows into infinity and remain unchanged. In both cases animas reached their individual maximum weight M_x, or M_{xx}. Another pattern of growth cessation is in the set with the phenotypes, which cannot reach their individual maximum weight nor the bifurcation point. In such animals the growth stops due to unknown causes. It is feasible to speculate that at a certain time point in ontogeny, such phenotypes slide on a growth trajectory in the form of a cycle. In this case, on the growth trajectory in the form of a limit cycle, a human's or animal's weight oscillates back and forth, around the centre point of the limit cycle. This pattern of weight dynamic is observed in humans and animals. In these phenotypes, a set with the centre points is feasible to consider as the set with the asymptotic maximum weights. One can conclude that under the model conditions two patterns of growth cessation in pigs are possible; one is determined by the dynamics of variable Z, while other is as yet unknown.

5. Conclusions

- The trajectory of growth of pigs is a discontinuous function. The function has a global minimum in the point Z = 2/5, it is discontinuous in the point Z = 1, and $Z = Z_x$. Animals can grow if Z > 2/3.
- It follows from the model that in the point $Z = Z_x$ a first-order phase transition takes place, and subsequent bifurcation of the growth trajectory shows up. In ontogeny, this sequence of events has a biological meaning.
- In the model, feed conversion coefficient Z is the order parameter. The growth stops and resumes contingent on the dynamics of Z. In animals' ontogeny, the two events a first-order phase transition, and subsequent bifurcation, developing in sequence, ensure transition to a new trajectory of growth. This two-stage process is to consider as a model of a pattern of the systemic regulation of growth in animals.
- During bifurcation, the four following growth phenotypes emerge: L1, L2, L3. And, heterozygote phenotype L12 has been brought about as a result of the coherent expression of L1, and L2 genes.
- Growth of animals is a dynamic process. In the study the growth of pigs was modelled as a dynamic system. The results of the model are comparable with the experimental data.

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