Analytical Solutions to Generalized Nonlinear Schrödinger Equation by Adomian Decomposition Technique

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

We study the higher-order nonlinear Schrödinger equation which takes care of the second as well as third order dispersion effects, cubic and quintic self phase modulations, self steepening and nonlinear dispersion effects. Taking advantage of the initial condition, we transform the previous equation into a nonlinear functional equation to which we apply a powerful analytical method called the Adomian decomposition method. We compute the Adomian's polynomials of corresponding infinite series solution. Assuming that the initial condition and all its derivatives converge to zero sufficiently rapidly as the time approaches to infinity, we established the convergence of the previous series. The last part of the paper describes applications resulting from nonlinear propagation phenomena in optical fibers. Numerical simulations are developed and it is further shown that comparison with other results yields a good qualitative agreement. These results demonstrate the robustness of the proposed method.

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1 INTRODUCTION

For describing the dynamics of light pulses in optical fibers, one uses the mathematical model namely the nonlinear Schrödinger (NLS) equation [1]. In [2], the NLS eqaution is formulated as a variational problem and comparison with scattering theory showed agreement. For picosecond light pulses, the NLS equation includes only the group velocity dispersion and the self-phase modulation, well known in fibers [3]. However, as one increases the intensity of the incident light power to produce shorter (femtosecond) pulses, additional nonlinear effects become important and the dynamics of pulses needs to be described in the frame of a generalized nonlinear Schrödinger equation (GNLSE) that includes higher-order nonlinear terms [4], [5]. Propagation of femtosecond pulse in fiber links can be described by the GNLSE which takes care of the second as well as third order dispersion effects, cubic and quintic self phase modulations, self steepening and nonlinear dispersion effects and can be written as [6] and [7] :

$$\psi_z + a_1\psi + a_2\psi_{tt} + a_3|\psi|^2\psi + a_4\psi_{ttt}$$
$$+a_5(|\psi|^2\psi)_t + a_6(|\psi|^4\psi)_t + a_7|\psi|^4\psi 0, \quad (1.1)$$

where ψ is a complex valued function of the normalized spatial coordinate z (distance of transmission through the fiber) and the normalized time t (ψ the slowly varying envelope of the electric field), $a_j, j = 1, \ldots, 7$ are complex parameters and the subscripts t and z denote the differentiation with respect to time and space, respectively ; ψ_{tt} represents the normalized group velocity dispersion, with the next cubic term describing self phase modulations due to the Kerr effect. The term involving a_4 represents the third order dispersion, while the term involving a_5 is the self steepening effect. The sixth and seventh terms in the left hand side respectively describe the nonlinear dispersion and self phase modulation effects due to quintic nonlinearity. The original input optical pulse is defined as

$$\psi(z,t)|_{z=0} = g(t), \quad (1.2)$$

where g is a given complex valued function of the time t.

The GNLSE has become a standard tool for simulating the propagation of strong light pulses in optical fibers [8], [9], [10]. For example, in order to describe the optical pulse propagation in a nonlinear dispersion medium, one derived this previous equation from Maxwell's equations under the slowly varying envelope approximation [11]. For this equation, both the analytical and the numerical solutions have been subject to intensive investigations. It is a well known fact that a few exact solutions of the GNLSE are known even now. This has been largely due to the complexity of the partial differential equation. The construction of an exact solution for the nonlinear evolution equation is an important topic in the study of nonlinear physical phenomena because exact solutions can help one to well understand the mechanism of the complicated physical phenomena and dynamical processes modeled by nonlinear evolution equations. Various methods have been employed for the analytical study, the Lagrangian variational method being the most common of them. In this method, the Lagrangian density of the system is constructed and a suitable trial function for the pulse is assumed. In a series of remarkable papers [6], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22] the authors have studied this equation using various approaches. These methods can be classified into two broad categories known as the finite-difference methods and the pseudo spectral methods [11]. Nowadays the method that has been used extensively is the split-step Fourier method which has a faster calculation speed than the other finite-difference methods. So it is very important to develop efficient methods to solve this equation. One of the methods is the Adomian decomposition method for solving a wide range of physical problems [23], [24]. Several modifications were improved its ability in [25] , [26], [24], [27], [28], [29], [30], [31], [7], [32]. An advantage of this method is that it can provide analytical

approximation or an approximated solution to a wide class of nonlinear equations without linearization, perturbation, closure approximation or discretization methods [33]. A recent work has been done by Biswas et al. [34] on optical solitons by using the Laplace-Adomian decomposition method. As a result numerical dispersive bright and dark optical solitons have been displayed and the precision obtained is excellent. The paper is organized as follows. In section 2, we solve the generalized nonlinear Schrödinger equation and prove the convergence of the solutions. In section 3, numerical investigation of the obtained analytical solutions is given. A brief conclusion is incorporated en section 4.

2 ANALYTICAL SOLUTION

In this section, we provide the analytical solution to the problem (1.1) and (1.2).
Taking account of the initial condition (1.2) and integrating the equation (1.1) from 0 to z, we have

$$\psi - N(\psi) = g(t) \tag{2.1}$$

where

$$N(\psi) = \int_0^z L\big[\psi(\xi,t), \psi^\star(\xi,t), \psi_t(\xi,t), \psi_t^\star(\xi,t), \psi_{tt}(\xi,t), \psi_{ttt}(\xi,t)\big] d\xi;$$

$$\begin{aligned}
L\big(\psi, \psi^\star, \psi_t, \psi_t^\star, \psi_{tt}, \psi_{ttt}\big) &= -a_1\psi - a_2\psi_{tt} - a_3\psi^2\psi^\star - a_4\psi_{ttt} - a_5\big(2\psi\psi_t\psi^\star + \psi^2\psi_t^\star\big) \\
&\quad -a_6\big(3\psi^2\psi_t\psi^{\star 2} + 2\psi^3\psi_t^\star\psi^\star\big) - a_7\psi^3\psi^{\star 2}.
\end{aligned}$$

Here N is a nonlinear operator from a Hilbert space $\mathcal{H}$ into $\mathcal{H}$.
In [23]-[24], Adomian decomposition method for solving linear and nonlinear equations was developed. In [27], [26] it was shown that the Adomian decomposition method provides a fast convergent series. We assume that (2.1) has a unique solution . Let us write the solution of (2.1) as an infinite series as follows

$$\psi = \sum_{n \geq 0} \psi_n \tag{2.2}$$

using the following scheme :

$$\begin{aligned}
\psi_0 &= g(t); \\
\psi_{n+1} &= A_n(\psi_0, \psi_2, \ldots, \psi_n), \quad n = 0, 1, 2, 3, \ldots,
\end{aligned}$$

where $A_n(\psi_0, \psi_1, \ldots, \psi_n)$ are polynomials of $\psi_0, \psi_1, \ldots, \psi_n$ defined by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i \geq 0} \lambda^i \psi_i\right)\right]_{\lambda=0}, n = 0, 1, 2, 3, \ldots$$

It comes that

$$A_n = \frac{1}{n!}\int_0^z \left[\frac{d^n}{d\lambda^n} L\left(\sum_{i \geq 0} \lambda^i \psi_i, \sum_{i \geq 0} \lambda^i \psi_i^\star, \sum_{i \geq 0} \lambda^i \psi_{it}, \sum_{i \geq 0} \lambda^i \psi_{it}^\star, \sum_{i \geq 0} \lambda^i \psi_{itt}, \sum_{i \geq 0} \lambda^i \psi_{ittt}\right)\right]_{\lambda=0} d\xi$$

Let us determine these polynomials. See Appendix for more details. We have

$$\psi_0 = g(t); \tag{2.3}$$

$$\psi_1 = zh_1(t); \tag{2.4}$$

$$\psi_2 = \frac{1}{1!}z^2 h_2(t); \tag{2.5}$$

$$\psi_3 = \frac{1}{2!}z^3 h_3(t) \tag{2.6}$$

and by induction

$$\psi_n \;=\; \frac{z^n h_n(t)}{(n-1)!}, \quad n \geq 1. \tag{2.7}$$

Therefore, the function ψ defined by

$$\psi(z,t) = g(t) + \sum_{n \geq 1} \frac{z^n h_n(t)}{(n-1)!} \tag{2.8}$$

is the exact solution of the previous equation satisfying the condition

$$\psi(z,t)|_{z=0} = g(t)$$

where h_i are functions depending on $g, g^\star$ and their derivatives, given by

$$
\begin{aligned}
h_1(t) \;=\;& -a_1 g(t) - a_2 g''(t) - a_3 |g(t)|^2 g(t) - a_4 g'''(t) - a_5 \big[2|g(t)|^2 g'(t) \\
& + g^2(t) g^{\star\prime}(t)\big] - a_6 \big[3g'(t)|g(t)|^4 + 2g^2(t)|g(t)|^2 g^{\star\prime}(t)\big] - a_7 |g(t)|^4 g(t);
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
h_2(t) \;=\;& \frac{1}{2}\Big\{ h_1(t)\Big[-a_1 - 2a_3|g(t)|^2 - 2a_5\big(g'(t)g^\star(t) + g(t)g^{\star\prime}(t)\big) \\
& -6a_6\big(|g(t)|^2 g'(t)g^\star(t) + g(t)|g(t)|^2 g^{\star\prime}(t)\big) - 3a_7|g(t)|^4\Big] \\
& + h_1^\star(t)\Big[-a_3 g^2(t) - 2a_5 g(t)g'(t) - a_6\big(6|g(t)|^2 g(t)g'(t) \\
& + 2g^3(t)g^{\star\prime}(t)\big) - 2a_7 g^2(t)|g(t)|^2\Big] + h_1'(t)\Big[-2a_5|g(t)|^2 - 3a_6|g(t)|^4\Big] \\
& + h_1^{\star\prime}(t)\Big[-a_5 g^2(t) - 2a_6 g^2(t)|g(t)|^2\Big] - a_2 h_1''(t) - a_4 h_1'''(t)\Big\};
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
h_3(t) \;=\;& \frac{1}{3}\Big\{ h_1^2(t)\big[-2a_3 g^\star(t) - 2a_5 g^{\star\prime}(t) - 6a_6\big(g'(t)g^{\star 2}(t) + 2|g(t)|^2 g^{\star\prime}(t)\big) \\
& -6a_7 g(t)g^{\star 2}(t)\big] + h_1^{\star 2}\big[-6a_6 g^2(t)g'(t) - 2a_7 g^3(t)\big] \\
& +2h_1(t)h_1^\star(t)\big[-2a_3 g(t) - 2a_5 g'(t) - 6a_6\big(2|g(t)|^2 g'(t) + g^2(t)g^{\star\prime}(t)\big) \\
& -6a_7 g(t)|g(t)|^2\big] + 2h_1(t)h_1'(t)\big(-2a_5 g^\star(t) - 6a_6|g(t)|^2 g^\star(t)\big) \\
& +2h_1(t)h_1^{\star\prime}(t)\big[-2a_5 g(t) - 6a_6 g(t)|g(t)|^2\big] + 2h_1^\star(t)h_1'(t)\big[-2a_5 g(t) \\
& -6a_6 g(t)|g(t)|^2\big] - 4a_6 g^3(t)h_1^\star(t)h_1^{\star\prime}(t) + 2h_2(t)\big[-a_1 - 2a_3|g(t)|^2 \\
& -2a_5\big(g'(t)g^\star(t)\big) + g(t)g^{\star\prime}(t) - 6a_6\big(|g(t)|^2 g'(t)g^\star(t) + |g(t)|^2 g(t)g^{\star\prime}(t)\big) \\
& -3a_7|g(t)|^4\big] + 2h_2^\star(t)\big[-a_3 g^2(t) - 2a_5 g(t)g'(t) - a_6\big(6|g(t)|^2 g(t)g'(t) \\
& +2g^3(t)g^{\star\prime}(t)\big) - 2a_7|g(t)|^2 g^2(t)\big] + 2h_2'(t)\big[-2a_5|g(t)|^2 - 3a_6|g(t)|^4\big] \\
& +2h_2^{\star\prime}(t)\big[-a_5 g^2(t) - 2a_6|g(t)|^2 g^2(t)\big] - 2a_2 h_2''(t) - 2a_4 h_2'''(t)\Big\}.
\end{aligned} \tag{2.11}
$$

Finally we arrive at the following result Assume that the functions $g, \; h_n$ satisfy the conditions

$$
\left\{
\begin{array}{l}
g(t) \longrightarrow 0 \quad \text{as} \quad |\mathrm{t}| \longrightarrow +\infty \\
\sup_{t \in \mathbf{R}} |h_n(t)| \leq c_n, \quad \forall \quad n \geq 1
\end{array}
\right. \tag{2.12}
$$

and the positive numerical series

$$\sum_{n \geq 1} c_n \tag{2.13}$$

converges.
Then the functional series

$$\sum_{n \geq 1} \frac{z^n h_n(t)}{(n-1)!} \qquad (2.14)$$

converges uniformly on the set $[0, +\infty[\times \mathbf{R}$. Indeed,

$$\sup_{t \in \mathbf{R}} \left| \sum_{n \geq 1} \frac{z^n h_n(t)}{(n-1)!} \right| \leq \sum_{n \geq 1} \frac{z^n}{(n-1)!} \sup_{t \in \mathbf{R}} |h_n(t)|.$$

By virtue of the conditions (2.12) and (2.13),

$$\sup_{t \in \mathbf{R}} \left| \sum_{n \geq 1} \frac{z^n h_n(t)}{(n-1)!} \right| \leq \sum_{n \geq 1} \frac{c_n z^n}{(n-1)!}$$

By means of d'Alembert's test, the series

$$\sum_{n \geq 1} \frac{c_n z^n}{(n-1)!}$$

converges. Therefore,

$$\sum_{n \geq 1} \frac{z^n h_n(t)}{(n-1)!}$$

converges uniformly on the set $[0, +\infty[\times \mathbf{R}$. $\square$

3 NUMERICAL APPROACH AND RESULTS

The purpose of this part is to provide numerical results for comparison with other results obtained using other methods of calculations both analytically and numerically. During the numerical calculation, different input sources with different duration, and different choices of parameters are used. As it is well known, the dispersion and nonlinear lengths characterize the pulse propagation in a given fiber. Physically, the dispersion length L_D is length at which a Gaussian pulse broadens by a factor of 2 due to group velocity dispersion. The dispersion length depends strongly on the half-width of the source slowly varying pulse envelope and the second order dispersion term and L_N the nonlinear length given by :

$$L_D = \frac{T_0^2}{\beta_2}, \qquad L_N = \frac{1}{\gamma P_0}; \qquad (3.1)$$

where P_0 is the peak power and γ is the nonlinear Kerr effect coefficient.

Some specific advantages of the results :
The solutions contain both the parameters of the medium and the pulse ;
The solutions provide infinite series approximation in an analytic form ;
The implementation is easy and the convergence of the solutions is very fast.

3.1 Case of Picosecond Pulses

We use the following parameters and compare our results with those in [7] where rapid numerical difference recurrent formula of nonlinear Schrödinger equation is used. i.e $T_0 = 20ps$, $\beta_2 = -4ps^2/km$, $\beta_3 = 0ps^3/km$, the fiber loss is equal to zero as well as the nonlinear Kerr effect parameter. The propagation distances are respectively : $z = 0km$, $z = 10km$, and $z = 16km$. The results presented in Fig. 1 and Fig. 2 provide the same Gaussian profiles as those presented in figure 1 in [7]. It is clearly observed that when the propagation distance increases the peak power decreases and the wave spreads while keeping its constant energy. This result

is expected and confirms the predictions. But unfortunately, what is not expected and constitute a limitation of our approach is that the distance propagation must remain small compared to the distance L_D. The validity of our method requires to restrict the propagation distance to the interval $[0, L_D]$. On the other side, the comparison is simply analytical and does not strictly take into account the numerical values of the output energy. In fact the initial idea was to compare numerical values of the energy but we arrived at results such that the propagation distances that are input parameters are not the same. We can conclude that the optical pulse propagation in optical medium can be obtained by Adomian decomposition method provided that the propagation distance is less than the dispersion length.

From Fig. 1 and Fig. 2 we can conclude that the shape of the pulse for a given propagation distance z depends on the input optical pulse. And as we have observed in Fig. 1, the peak

power of the pulse at the input is high and as the pulse propagates in the fiber, the peak decreases and it spreads. We also restrict ourselves in the propagation interval to $[0, L_D]$.

3.2 Case of Femtosecond Pulses

We consider that the input optical pulse is hyperbolic secant, the dispersive parameters used are the same as those in [20] : $\beta_2 = -1ps^2/km$, $\beta_3 = 0.0012ps^3/km$.
In Fig. 3 it can be observed that the pulse is shifted to the positive time axis during its propagation. In the numerical simulations, it is observed that the third order dispersion parameter and the self steepening parameter influence strongly the shape of the pulse during propagation. In Fig. 4 the influence of the fiber loss parameter a_1 is shown. The optical pulse energy dropped significantly. The arbitrary parameters used in Fig. 4 are : $a_2 = -i/2$, $a_3 = -i$, $a_4 = 0.02$, $a_5 = 0.8$, $a_6 = 0.1$, $a_7 = 0.5$.
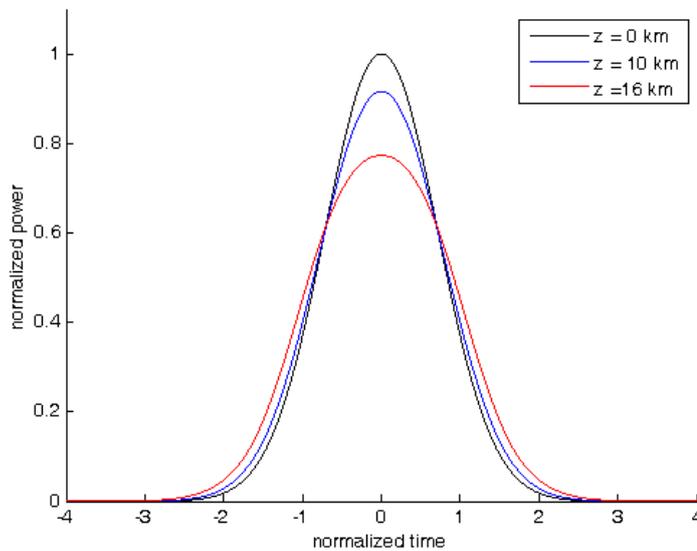
FIGURE 1 − Adomian analytical solutions plots for distance propagation z=0 km, z=10 km and z=16 km respectively. The input optical pulse is Gaussian without chirp and loss.
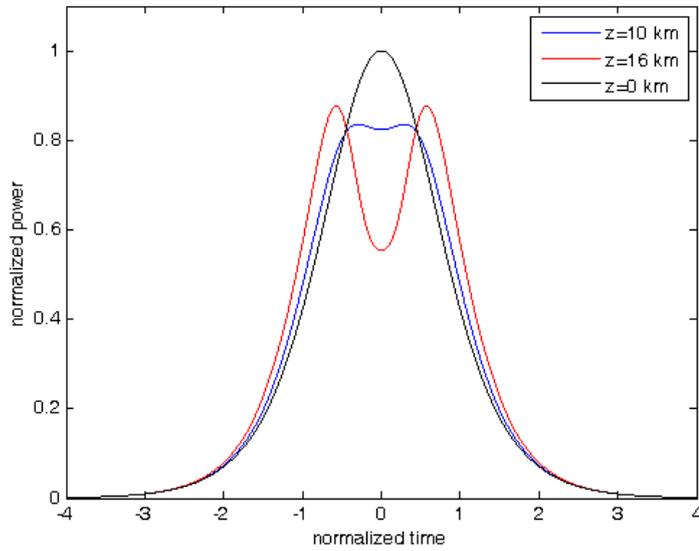
FIGURE 2 — Adomian analytical solutions plots for distance propagation z=0 km, z=10 km and z=16 km respectively. The input optical pulse is hyperbolic secant without chirp and loss.
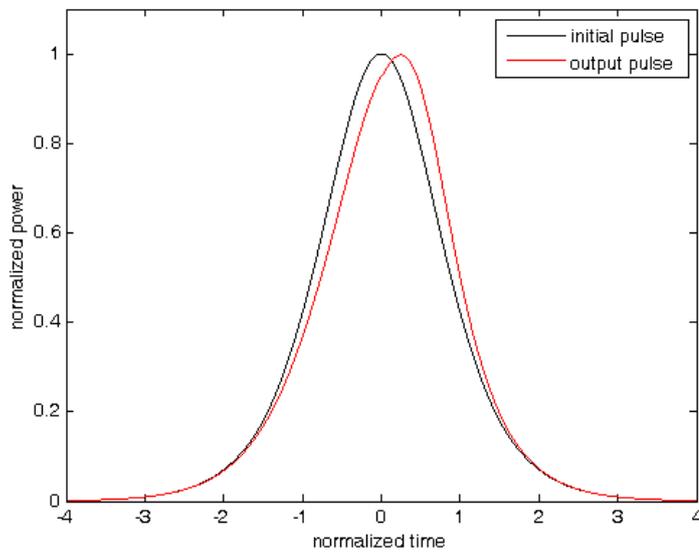
FIGURE 3 — Adomian analytical solutions plots for distance propagation $z = 10^{-5} km$. The input optical pulse is hyperbolic secant.
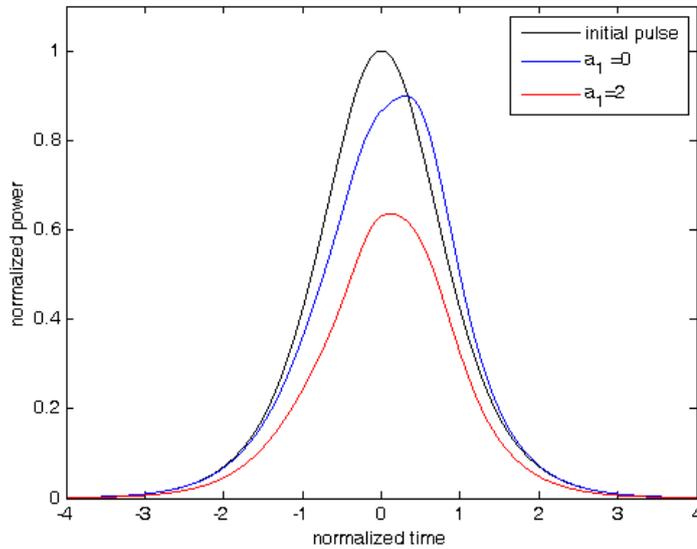
FIGURE 4 − Adomian analytical solutions plots for distance propagation $z = 10^{-5}km$. The input optical pulse is hyperbolic secant ; $\beta_2 = -1ps^2/km$, $\beta_3 = 0.0012$; $a_1 = 0$; $a_1 = 2$
.

4 CONCLUDING REMARKS

In this paper, we have investigated the analytical solutions for the generalized nonlinear Schrödinger equation using the Adomian decomposition method. For the optical input pulse converging to zero sufficiently rapidly as the time approaches to infinity and for the bounded functions depending on the optical input pulse and its derivatives, we proved that the obtained solutions series of Adomian's polynomials converges uniformly on a given set. In order to validate the availability of our approach, we simulate numerically these obtained solutions by studying propagation of optical pulses inside a nonlinear dispersive medium. The picosecond and femtosecond optical input pulses with Gaussian and hyperbolic secant profiles are used. Thus the results showed an acceptable physical behavior of the solutions when the propagation distance is less than the dispersive length. We can conclude that Adomian decomposition method is an useful and important tool for scrutinizing the inside of an optical fiber provided that the propagation distance is less than the dispersive length.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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APPENDIX

In this appendix we list the first and second order derivatives of L with respect to $\psi, \psi^\star$, $\psi_t, \psi_t^\star, \psi_{tt}, \psi_{ttt}$ defining the expressions of $\dfrac{dL}{d\lambda}$ and $\dfrac{d^2L}{d\lambda^2}$:

$$\frac{dL}{d\lambda} = \frac{\partial L}{\partial \psi}\frac{d\psi}{d\lambda} + \frac{\partial L}{\partial \psi^\star}\frac{d\psi^\star}{d\lambda} + \frac{\partial L}{\partial \psi_t}\frac{d\psi_t}{d\lambda} + \frac{\partial L}{\partial \psi_t^\star}\frac{d\psi_t^\star}{d\lambda} + \frac{\partial L}{\partial \psi_{tt}}\frac{d\psi_{tt}}{d\lambda} + \frac{\partial L}{\partial \psi_{ttt}}\frac{d\psi_{ttt}}{d\lambda}$$

$$\begin{aligned}
\frac{d^2L}{d\lambda^2} = \ & \frac{\partial^2 L}{\partial \psi^2}\Big(\frac{d\psi}{d\lambda}\Big)^2 + \frac{\partial^2 L}{\partial \psi^{\star 2}}\Big(\frac{d\psi^\star}{d\lambda}\Big)^2 \\
& + 2\frac{\partial^2 L}{\partial \psi \partial \psi^\star}\frac{d\psi}{d\lambda}\frac{d\psi^\star}{d\lambda} + 2\frac{\partial^2 L}{\partial \psi \partial \psi_t}\frac{d\psi}{d\lambda}\frac{d\psi_t}{d\lambda} + 2\frac{\partial^2 L}{\partial \psi \partial \psi_t^\star}\frac{d\psi}{d\lambda}\frac{d\psi_t^\star}{d\lambda} \\
& + 2\frac{\partial^2 L}{\partial \psi^\star \partial \psi_t}\frac{d\psi^\star}{d\lambda}\frac{d\psi_t}{d\lambda} + 2\frac{\partial^2 L}{\partial \psi^\star \partial \psi_t^\star}\frac{d\psi^\star}{d\lambda}\frac{d\psi_t^\star}{d\lambda} + \frac{\partial L}{\partial \psi}\frac{d^2\psi}{d\lambda^2} \\
& + \frac{\partial L}{\partial \psi^\star}\frac{d^2\psi^\star}{d\lambda^2} + \frac{\partial L}{\partial \psi_t}\frac{d^2\psi_t}{d\lambda^2} + \frac{\partial L}{\partial \psi_t^\star}\frac{d^2\psi_t^\star}{d\lambda^2} + \frac{\partial L}{\partial \psi_{tt}}\frac{d^2\psi_{tt}}{d\lambda^2} + \frac{\partial L}{\partial \psi_{ttt}}\frac{d^2\psi_{ttt}}{d\lambda^2}
\end{aligned}$$

with

$$\frac{\partial L}{\partial \psi} = -a_1 - 2a_3\psi\psi^\star - 2a_5\big(\psi_t\psi^\star + \psi\psi_t^\star\big) - 6a_6\big(\psi\psi_t\psi^{\star 2} + \psi^2\psi_t^\star\psi^\star\big) - 3a_7\psi^2\psi^{\star 2},$$

$$\frac{\partial L}{\partial \psi^\star} = -a_3\psi^2 - 2a_5\psi\psi_t - a_6\big(6\psi^2\psi_t\psi^\star + 2\psi^3\psi_t^\star\big) - 2a_7\psi^3\psi^\star,$$

$$\frac{\partial L}{\partial \psi_t} = -2a_5\psi\psi^\star - 3a_6\psi^2\psi^{\star 2}; \quad \frac{\partial L}{\partial \psi_t^\star} = -a_5\psi^2 - 2a_6\psi^3\psi^\star,$$

$$\frac{\partial L}{\partial \psi_{tt}} = -a_2; \quad \frac{\partial L}{\partial \psi_{ttt}} = -a_4.$$

$$\frac{\partial^2 L}{\partial \psi_t^2} = 0; \quad \frac{\partial^2 L}{\partial \psi_t^{\star 2}} = 0; \quad \frac{\partial^2 L}{\partial \psi_{tt}^2} = 0; \quad \frac{\partial^2 L}{\partial \psi_{ttt}^2} = 0;$$

$$\frac{\partial^2 L}{\partial \psi \partial \psi_{tt}} = 0; \quad \frac{\partial^2 L}{\partial \psi \partial \psi_{ttt}} = 0; \quad \frac{\partial^2 L}{\partial \psi^\star \partial \psi_{tt}} = 0; \quad \frac{\partial^2 L}{\partial \psi^\star \partial \psi_{ttt}} = 0; \quad \frac{\partial^2 L}{\partial \psi_t \partial \psi_t^\star} = 0;$$

$$\frac{\partial^2 L}{\partial \psi_t \partial \psi_{tt}} = 0; \quad \frac{\partial^2 L}{\partial \psi_t \partial \psi_{ttt}} = 0; \quad \frac{\partial^2 L}{\partial \psi_t^\star \partial \psi_{tt}} = 0; \quad \frac{\partial^2 L}{\partial \psi_t^\star \partial \psi_{ttt}} = 0; \quad \frac{\partial^2 L}{\partial \psi_{tt} \partial \psi_{ttt}} = 0.$$

$$\frac{\partial^2 L}{\partial \psi^2} = -2a_3\psi^\star - 2a_5\psi_t^\star - 6a_6\Big(\psi_t\psi^{\star 2} + 2\psi\psi_t^\star\psi^\star\Big) - 6a_7\psi\psi^{\star 2};$$

$$\frac{\partial^2 L}{\partial \psi^{\star 2}} = -6a_6\psi^2\psi_t - 2a_7\psi^3; \quad \frac{\partial^2 L}{\partial \psi \partial \psi_t} = -2a_5\psi^\star - 6a_6\psi\psi^{\star 2};$$

$$\frac{\partial^2 L}{\partial \psi \partial \psi^\star} = -2a_3\psi - 2a_5\psi_t - 6a_6\Big(2\psi\psi_t\psi^\star + \psi^2\psi_t^\star\Big) - 6a_7\psi^2\psi^\star;$$

$$\frac{\partial^2 L}{\partial \psi \partial \psi_t^\star} = -2a_5\psi - 6a_6\psi^2\psi^\star; \quad \frac{\partial^2 L}{\partial \psi^\star \partial \psi_t} = -2a_5\psi - 6a_6\psi^2\psi^\star; \quad \frac{\partial^2 L}{\partial \psi^\star \partial \psi_t^\star} = -2a_6\psi^3.$$